

ON THE LOWER BOUND FOR THE NUMBER OF FACETS OF A k -NEIGHBORLY POLYTOPE

ALEKSANDR MAKSIMENKO

ABSTRACT. We pose the conjecture that the number of facets $f_{d-1}(P)$ of a k -neighborly d -polytope P cannot be less than the number of its vertices $f_0(P)$ for $k \geq 2$. We prove that the statement of the conjecture is true in two cases: 1) when $d \leq 2k + 2\lfloor k/2 \rfloor$, and 2) when $f_0(P) \leq d + 1 + k^2$. For the case $f_0(P) = d + 3$, we have found the tight lower bound

$$f_{d-1}(P) - f_0(P) \geq \begin{cases} (k+2)(k^2+k-3)/3 & \text{if } k \in \{2, 3\}, \\ 2(k^2-1) & \text{if } k \geq 4. \end{cases}$$

Let P be a d -polytope, i.e., a d -dimensional convex polytope. An i -dimensional face of P is called *i-face*, 0-faces are *vertices*, 1-faces are *edges*, $(d-1)$ -faces are *facets*, and $(d-2)$ -faces are *ridges*. Let $f_i(P)$ be the number of i -faces of P , $0 \leq i \leq d-1$. The problem of estimating $f_i(P)$ (where P belongs to some class of polytopes) in terms of $f_0(P)$ is well known. For the class of simplicial polytopes, the problem is known as the upper bound and the lower bound theorems (see [6] for details). In particular [1],

$$(1) \quad f_{d-1}(P) \geq (d-1)(f_0(P) - d) + 2 \quad \text{for a simplicial } d\text{-polytope } P.$$

In 1990, G. Blind and R. Blind [2] solved the upper bound problem for the class of polytopes without a triangle 2-face. We raise the question for the class of 2-neighborly polytopes.

A d -polytope P is called *k -neighborly* if each subset of k vertices forms the vertex set of some face of P . Since every d -polytope is 1-neighborly, we will consider k -neighborly polytopes only for nontrivial cases $k \geq 2$. For $d < 2k$, there is only one combinatorial type of k -neighborly d -polytope. It is a d -simplex [6]. The same is true for $f_0(P) = d + 1$. Therefore, we suppose $d \geq 2k$ and $f_0(P) > d + 1$.

A $\lfloor d/2 \rfloor$ -neighborly polytope is called *neighborly*. In particular, every neighborly d -polytope is 2-neighborly for $d \geq 4$. The family of neighborly polytopes are investigated very intensively (see, e.g., [6]). For $d = 2k$, they have the maximum number of facets over all d -polytopes with n vertices [10]:

$$(2) \quad f_{d-1}(P_{\text{neighborly}}) = \frac{n}{n-k} \binom{n-k}{k}.$$

There exists a widespread feeling that k -neighborly polytopes are very common among convex polytopes [6, 4, 5]. Moreover, they appear as faces (with superpolynomial number of vertices) of combinatorial polytopes associated with NP-complete problems [7, 8].

As a reference point for further investigations we pose the following conjecture.

Supported by the project No. 477 of P. G. Demidov Yaroslavl State University within State Assignment for Research.

Conjecture 1. *The number of facets $f_{d-1}(P)$ of a k -neighborly d -polytope P cannot be less than the number of its vertices $f_0(P)$ for $k \geq 2$.*

In the section 1, we consider examples of 2-neighborly d -polytopes with

$$f_{d-1}(P) - f_0(P) < \frac{f_0(P)(f_0(P) - d - 1)}{0.4d}.$$

From (1), it follows that the conjecture is true if P is a simplicial polytope. The case $d = 2k$ is covered by (2). In the section 2, we prove the statement of the conjecture for the cases $d \leq 2k + 2\lfloor k/2 \rfloor$. In the section 3, the inequality $f_{d-1}(P) \geq d + k^2 + 1$ is proved for a k -neighborly d -polytope P with $f_0(P) \geq d + 2$. In the last section, we give the proof of the following theorem.

Theorem 1. *Let P be a k -neighborly d -polytope with $f_0(P) = d + 3$ and let $\Delta_3(k) = \min_P \{f_{d-1}(P) - f_0(P)\}$, then*

$$\Delta_3(k) = \begin{cases} (k+2)(k^2+k-3)/3 & \text{if } k \in \{2, 3\}, \\ 2(k^2-1) & \text{if } k \geq 4. \end{cases}$$

1. 2-NEIGHBORLY POLYTOPES WITH “SMALL” NUMBER OF FACETS

It is natural to try to construct examples of a 2-neighborly d -polytopes with as small as possible difference between facets and vertices. Here we consider two operations on polytopes. By these operations, we can construct new 2-neighborly polytopes without increasing the difference $f_{d-1}(P) - f_0(P)$.

From (2), we see that a 2-neighborly 4-polytope P with n vertices has exactly $n(n-3)/2$ facets. Thus

$$f_{d-1}(P) - f_0(P) = f_0(P)(f_0(P) - d - 1)/2, \quad d = 4.$$

By constructing a pyramid with the basis P , we get a 2-neighborly 5-polytope Q with $f_0(Q) = f_0(P) + 1$ and $f_4(Q) = f_3(P) + 1$. Repeating this procedure, we can construct an example of a 2-neighborly d -polytope Q with

$$(3) \quad f_{d-1}(P) - f_0(P) = f_0(P)(f_0(P) - 5)/2.$$

The other good operation is a *join* of two polytopes [4]:

$$P * P' := \text{conv} \left(\{(x, 0, 0) \in \mathbb{R}^{d+d'+1} \mid x \in P\} \cup \{(0, y, 1) \in \mathbb{R}^{d+d'+1} \mid y \in P'\} \right),$$

where P is a d -polytope and P' is a d' -polytope. $P * P'$ has dimension $d + d' + 1$, $f_0(P) + f_0(P')$ vertices, $f_{d-1}(P) + f_{d'-1}(P')$ facets [4]. Moreover, if P and P' are k -neighborly, then $P * P'$ is also k -neighborly.

Let P_0 be a 2-neighborly 4-polytope with n vertices, $n \geq 5$. Hence $P_1 = P_0 * P_0$ is 2-neighborly 9-polytope with $2n$ vertices and $n(n-3)$ facets. Let us define P_m recursively:

$$P_{m+1} = P_m * P_m, \quad m \in \mathbb{N}.$$

Thus P_m is a 2-neighborly d -polytope with

$$d = 5 \cdot 2^m - 1, \quad f_0(P_m) = 2^m n, \quad f_{d-1}(P_m) = 2^{m-1} n(n-3).$$

Therefore,

$$f_{d-1}(P_m) - f_0(P_m) = \frac{f_0(P_m)(f_0(P_m) - d - 1)}{2^{m+1}} < \frac{f_0(P_m)(f_0(P_m) - d - 1)}{0.4d}.$$

This difference has a bit better assymptotic than (3).

2. SMALL DIMENSIONS

In this section we use the well known fact that all $(2k-1)$ -faces of a k -neighborly polytope are simplexes [6, Sec. 7.1].

Lemma 2. *If all i -faces of a d -polytope P are simplexes and $i \geq d/2$, then $f_i(P) \geq f_{d-i-1}(P)$.*

Proof. Let us count incidences between j -faces and $(j-1)$ -faces of P , $0 < j \leq i$. Note that every $(j-1)$ -face of a d -polytope is incident with at least $(d-j+1)$ j -faces [6]. Hence,

$$(d-j+1)f_{j-1}(P) \leq \sum_{y \in F_j(P)} f_{j-1}(y),$$

where $F_j(P)$ is the set of all j -faces of P , $f_{j-1}(y)$ is the number of $(j-1)$ -faces of y . Since j -faces are simplexes,

$$(4) \quad (d-j+1)f_{j-1}(P) \leq (j+1)f_j(P) \quad \text{for } 0 < j \leq i.$$

Let d be even, $d = 2m$, $m \in \mathbb{N}$. Let $j = m$. From inequality (4), we get

$$(5) \quad (m+1)f_{m-1}(P) \leq (m+1)f_m(P).$$

Suppose that $i > m$. Hence, substituting $j \in \{m-1, m+1\}$ in (4), we obtain

$$(m+2)f_{m-2}(P) \leq mf_{m-1}(P) \quad \text{and} \quad mf_m(P) \leq (m+2)f_{m+1}(P).$$

Combining this with (5), we have

$$f_{m-2}(P) \leq f_{m+1}(P).$$

By repeating this procedure, it is easy to get

$$f_{d-i-1}(P) \leq f_i(P) \quad \text{for even } d.$$

The case $d = 2m-1$, $m \in \mathbb{N}$, are proved by analogy. \square

Theorem 3. *If P is a k -neighborly d -polytope and $d \leq 2k+2\lfloor k/2 \rfloor$, then $f_{d-1}(P) \geq f_0(P)$.*

Proof. First let us suppose that

$$d \leq 3k-1.$$

Recall that every i -face of a k -neighborly d -polytope P is a simplex for $i < 2k$ [6, Sec. 7.1]. Using Lemma 2, we get

$$(6) \quad f_{d-k}(P) \geq f_{k-1}(P).$$

Note also that for a k -neighborly d -polytope P we can use the implication

$$(7) \quad f_{k-1}(P) \leq f_{d-k}(P) \Rightarrow f_0(P) \leq f_{d-1}(P).$$

Indeed, $f_{k-1}(P) = \binom{f_0(P)}{k}$ and $f_{k-1}(P) \geq f_{k-1}(Q)$ for any d -polytope Q with $f_0(Q) = f_0(P)$. By using duality, we get $f_{d-k}(P) \leq \binom{f_{d-1}(P)}{k}$. Finally note that

$$\binom{f_0(P)}{k} \leq \binom{f_{d-1}(P)}{k} \Rightarrow f_0(P) \leq f_{d-1}(P).$$

Combining (6) and (7), we obtain

$$f_0(P) \leq f_{d-1}(P) \quad \text{for } d \leq 3k-1.$$

Now let us suppose that $k = 2m$, $m \in \mathbb{N}$, and $d = 3k$. Until the end of the proof we use the shorthand $f_i := f_i(P)$.

From Euler's equation [6]

$$\sum_{i=0}^{d-1} (-1)^i f_i = 0,$$

we get

$$\sum_{i=0}^{k-1} (-1)^i f_i + \sum_{i=d-k}^{d-1} (-1)^i f_i = (f_{d-k-1} - f_k) + \cdots + (f_{d/2} - f_{d/2-1}).$$

Using Lemma 2, we have

$$(f_{d-k-1} - f_k) + \cdots + (f_{d/2} - f_{d/2-1}) \geq 0$$

and

$$\sum_{i=0}^{k-1} (-1)^i f_i + \sum_{i=d-k}^{d-1} (-1)^i f_i \geq 0.$$

That is

$$(8) \quad f_{k-1} - f_{k-2} + \cdots + f_1 - f_0 \leq f_{d-k} - f_{d-k+1} + \cdots + f_{d-2} - f_{d-1}.$$

Note that

$$f_i = \binom{f_0}{i+1}, \quad 0 \leq i < k,$$

for a k -neighborly polytope. Let

$$D_k(n) = \sum_{i=1}^k (-1)^i \binom{n}{i}, \quad n \geq 2k.$$

Thus,

$$(9) \quad f_{k-1} - f_{k-2} + \cdots + f_1 - f_0 = D_k(f_0).$$

Now we prove that

$$(10) \quad f_{k-1}(Q) - f_{k-2}(Q) + \cdots + f_1(Q) - f_0(Q) \leq D_k(f_0(Q))$$

for any d -polytope Q with $d \geq k$. Since $f_i(Q) \leq \binom{f_0(Q)}{i+1}$, $0 \leq i < k$, it is sufficient to show $\frac{f_i(Q)}{f_{i-1}(Q)} \leq \frac{f_0(Q)-i}{i+1}$. Note that every $(i-1)$ -face is incident with no more, than $(f_0 - i)$ i -faces. (Because every such i -face has at least one unique vertex that does not belong to other such i -faces.) Note also that every i -face has at least $i+1$ $(i-1)$ -faces. Thus, $(f_0(Q) - i)f_{i-1}(Q) \geq (i+1)f_i(Q)$.

By duality, (10) is transformed to

$$f_{d-k} - f_{d-k+1} + \cdots + f_{d-2} - f_{d-1} \leq D_k(f_{d-1}),$$

which is true for any d -polytope, $d \geq k$. In particular, it is true for a k -neighborly d -polytope P . Combining this with (8) and (9), we obtain $D_k(f_0) \leq D_k(f_{d-1})$. Since $D_k(n)$ is strictly increasing function, we get $f_0 \leq f_{d-1}$. \square

3. SIMPLICIAL POLYTOPES AND POLYTOPES WITH SMALL NUMBER OF VERTICES

The tight lower bound for the number of facets of a simplicial k -neighborly d -polytope can be found by using g -theorem.

Let $M_d = (m_{i,j})$ be $(\lfloor d/2 \rfloor + 1) \times (d + 1)$ -matrix with entries

$$m_{i,j} = \binom{d+1-i}{d+1-j} - \binom{i}{d+1-j}, \quad 0 \leq i \leq \lfloor d/2 \rfloor, \quad 0 \leq j \leq d.$$

The matrix M_d has nonnegative entries and the left $(\lfloor d/2 \rfloor + 1) \times (\lfloor d/2 \rfloor + 1)$ -submatrix is upper triangular with ones on the main diagonal. The g -theorem states that the f -vector $f = (f_{-1}, f_0, f_1, \dots, f_{d-1})$ of a simplicial d -polytope is equal to gM_d where g -vector $g = (g_0, g_1, \dots, g_{\lfloor d/2 \rfloor})$ is an M -sequence (see, e.g., [11]).

Theorem 4. *If P is a simplicial k -neighborly d -polytope ($k \leq d/2$) with n vertices and $n \geq d + 2$, then*

$$f_j(P) \geq \sum_{i=0}^k \left(\binom{d+1-i}{d-j} - \binom{i}{d-j} \right) \binom{n-d-2+i}{i}.$$

In particular,

$$f_{d-1}(P) \geq \sum_{i=0}^k (d+1-2i) \binom{n-d-2+i}{i}.$$

Proof. From the equality $f = gM_d$, we can evaluate the first $k + 1$ entries of g :

$$(11) \quad \begin{aligned} g_0 &= f_{-1}(P) = 1, \\ g_0 m_{0,1} + g_1 &= f_0(P) = n, \\ g_0 m_{0,2} + g_1 m_{1,2} + g_2 &= f_1(P) = \binom{n}{2}, \\ &\dots \\ g_0 m_{0,k} + g_1 m_{1,k} + \dots + g_k &= f_{k-1}(P) = \binom{n}{k}. \end{aligned}$$

We suppose that $k \leq d/2$ and $n \geq d + 2$. By using induction on j , we prove

$$(12) \quad g_j = \binom{n-d-2+j}{j}, \quad 0 \leq j \leq k.$$

Obviously, $g_0 = 1$ and $g_1 = n - d - 1$. Suppose that the equality (12) is true for $j = l$, $1 \leq l \leq k - 1$. From (11), we have

$$\sum_{i=0}^l \binom{n-d-2+i}{i} \binom{d+1-i}{d-l} + g_{l+1} = \binom{n}{l+1}.$$

Recall one of the formulation of Vandermonde's convolution:

$$\sum_i \binom{\alpha+i}{i} \binom{\beta-i}{\gamma-i} = \binom{\alpha+\beta+1}{\gamma}.$$

Hence, $g_{l+1} = \binom{n-d-1+l}{l+1}$.

By g -theorem,

$$f_j = \sum_{i=0}^{\lfloor d/2 \rfloor} g_i m_{i,j+1}.$$

Since g is an M -sequence, we may assume $g_i = 0$ for $i > k$. Hence,

$$f_j \geq \sum_{i=0}^k g_i m_{i,j-1} = \sum_{i=0}^k \binom{n-d-2+i}{i} m_{i,j+1}.$$

□

Corollary 5. *If P is a simplicial k -neighborly d -polytope with $f_0(P) \geq d+2$, then $f_{d-1}(P) \geq d+k^2+1$.*

Corollary 6. *If P is a simplicial k -neighborly d -polytope with $f_0(P) = d+3$, then $f_{d-1}(P) \geq \frac{(k+1)(k+2)(3d+3-4k)}{6}$.*

Now it is not difficult to prove the following theorem.

Theorem 7. *If P is a k -neighborly d -polytope that is not a simplex, then*

$$(13) \quad f_{d-1}(P) \geq d+1+k^2.$$

Proof. The proof is by induction over d . For $d = 2k$, the validity of (13) immediately follows from (2).

Suppose that (13) is true for $d = m$, $m \geq 2k$. Let P be a k -neighborly $(m+1)$ -polytope. If P is simplicial, then the inequality (13) follows from Corollary 5. Now suppose that P has a nonsimplicial facet Q . Hence Q is a k -neighborly m -polytope with $f_0(Q) \geq m+2$ and the inequality (13) holds for it, by the induction hypothesis. Therefore,

$$f_m(P) \geq f_{m-1}(Q) + 1 \geq m+1+k^2+1.$$

□

4. THE CASE $f_0(P) = \dim(P) + 3$

In this section we consider only d -polytopes with exactly $d+3$ vertices. For proving Theorem 1, we use reduced Gale diagrams.

4.1. Reduced Gale diagrams. A *reduced Gale diagram* of a polytope P consists of points in \mathbb{R}^2 , placed at the center O and the vertices of a regular $2n$ -gon (of radius 1), $n \geq 2$. For the sake of convenience, we enumerate the vertices of the $2n$ -gon by numbers from 0 to $2n-1$ going clockwise. (It does not matter what point will be the first.) The $2n$ points have nonnegative integer labels (multiplicities) $m_0, m_1, \dots, m_{2n-1}$. The center O has label $m(O)$. In the following we assume

$$m_i \stackrel{\text{def}}{=} m_i \bmod 2n.$$

In particular, $m_{i+2n} \stackrel{\text{def}}{=} m_{i-2n} \stackrel{\text{def}}{=} m_i$. A pair of opposite labels m_i and m_{i+n} , $i \in [n]$, is called a *diameter*. Hereinafter, n is always the number of diameters of a reduced Gale diagram. These labels have the following properties (see [6, Sec. 6.3] and [3]):

P1: $m(O) + m_1 + \dots + m_{2n-1} = d+3$. (The sum of all labels equals the number of vertices of P .)

P2: $m_i + m_{i+n} > 0$ for every $i \in [n]$. (The sum of the labels of a diameter is not equal to 0. Otherwise, we can construct this diagram on a $(2n-2)$ -gon.)

P3: $m_{i-1} + m_i > 0$ for every $i \in [2n]$. (Two neighbour vertices of the $2n$ -gon cannot both have label 0. Otherwise the appropriate two diameters can be glued and we get a $(2n-2)$ -gon.)

P4: $\sum_{j=i+1}^{i+n-1} m_j \geq 2$ for every $i \in [2n]$. (The sum of the labels in any open semicircle is at least 2.)

A subset S of these $d+3$ points¹ called a *cofacet* if the convex hull of S is a simplex with the center O in its relative interior and $\text{ext}(S) = \text{ext}(\text{conv}(S))$ (the set of vertices of $\text{conv}(S)$ coincides with S). Therefore, there are only three types of cofacets: the center O itself, two opposite vertices of the $2n$ -gon, three vertices of the $2n$ -gon that form a triangle with O in its interior.

Theorem 8. [6, Sec. 6.3] *There are one-to-one correspondence between combinatorial types of d -polytopes with $d+3$ vertices and reduced Gale diagrams with the properties P1–P4. Moreover,*

F: *The number of facets of a polytope is equal to the number of cofacets of the appropriate reduced Gale diagram.*

S: *An appropriate polytope is simplicial iff $m(O) = 0$ and $m_i m_{n+i} = 0 \forall i \in [n]$.*

N: *An appropriate polytope is k -neighborly iff $\sum_{j=i+1}^{i+n-1} m_j \geq k+1$ for every $i \in [2n]$. (The sum of the labels in any open semicircle is at least $k+1$.) Such a diagram is called k -neighborly.*

Note that the properties P2 and P3 always may be obtained by making *standard operations*:

D: Delete the labels with $m_i = m_{n+i} = 0$ and reduce the number of diameters.

G: If $m_i = m_{i+1} = 0$, then glue m_i and m_{i+1} , m_{i+n} and m_{i+1+n} , and reduce the number of diameters.

The standard operations do not affect the k -neighborliness and the number of cofacets. So, in the following we will not take properties P2 and P3 into account.

Note also that changing of the value of the label $m(O)$ does not affect the properties P2, P3, P4, and N, and does not change the difference $f_{d-1}(P) - f_0(P)$ of the appropriate polytope P . Thus, without loss of generality, we assume $m(O) = 0$.

We will say that a k -neighborly reduced Gale diagram is *extremal* if the difference between the number of cofacets and the sum of labels is minimal among all k -neighborly reduced Gale diagrams. In other words, the difference is equal to $\Delta_3(k)$ in Theorem 1.

Turn here to examples.

Example 1. Let $n = 2$ and $m_i = k+1$, $i \in [2n]$. Fig. 1a shows such a diagram for $k = 2$. The number of vertices of an appropriate k -neighborly polytope is equal to $4(k+1)$, the number of facets equals $2(k+1)^2$, and the difference is $2(k^2 - 1)$.

Example 2. Let $n = k+2$ and all the labels are equal to 1 (see fig. 1b). Hence, the number of vertices equals $2k+4$, the number of facets equals $2\binom{k+2}{3} + k+2$, and the difference is $(k+2)(k^2+k-3)/3$. Note that the difference is less than in the example 1 for $k \leq 3$.

Example 3. Let $n = 2k+3$, $m_{2i} = 0$, and $m_{2i-1} = 1$ for $i \in [n]$. Fig. 1c shows the case $k = 2$. An appropriate k -neighborly polytope is a simplicial one. It has dimension $2k$, $2k+3$ vertices and $(2k+3)(k+2)(k+1)/6$ facets. The difference

¹take into account the multiplicities

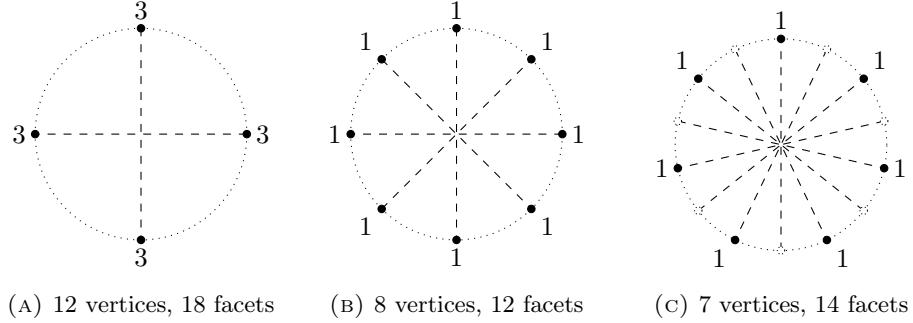


FIGURE 1. Reduced Gale diagrams for some 2-neighborly polytopes

between facets and vertices is $(2k+3)(k+4)(k-1)/6$ and it is greater than in the previous examples for every $k \geq 2$. Hence this diagram is not extremal.

Remark 9. From Corollary 6, it follows that $f_{d-1}(P) - f_0(P) > 2(k^2 - 1)$ for a simplicial k -neighborly d -polytope P with $f_0(P) = d+3$ and $k \geq 2$. Note that $f_{d-1}(P) - f_0(P) = 2(k^2 - 1)$ in the example 1. Therefore, an extremal k -neighborly diagram must be nonsimplicial.

A k -neighborly reduced Gale diagram is called *minimal* if reducing (decreasing) any of the labels m_i violates the condition N. It is easy to prove that the minimum for the difference between the number of cofacets and the sum of labels is attained on a minimal diagram. Thus, below we consider only minimal diagrams.

Let us note that $m_i \leq k+1$, $i \in [2n]$, for a minimal k -neighborly reduced Gale diagram. In the case $n=2$, the only minimal k -neighborly reduced Gale diagram has labels $m_i = k+1$, $i \in [2n]$ (see example 1).

Theorem 10 ([9]). *The sum of labels of any minimal k -neighborly reduced Gale diagram is not greater than $4(k+1)$. If the sum equals $4(k+1)$, then such Gale diagram is a 4-gon with labels $m_1 = m_2 = m_3 = m_4 = k+1$.*

The proof of Theorem 1 consists of the following steps. First of all, we show that $m_i + m_{i-1} \geq 2$, $i \in [2n]$, for an extremal Gale diagram (Lemmas 11–13). It gives us possibility to enumearate all extremal Gale diagrams for $k \leq 6$ with the help of a computer (Proposition 16). The case $k \geq 6$ is analyzed in the subsection 4.3.

4.2. Local properties of extremal Gale diagrams.

Lemma 11. *Let m_i be a positive label in a k -neighborly reduced Gale diagram D with n diameters. In addition, let $m_{i+n} = 0$ and $m_{i+n-1} > 0$ (see fig. 2). Then the displace operation*

$$m_i := m_i - 1 \quad \text{and} \quad m_{i-1} := m_{i-1} + 1$$

reduces the total number of facets by at least k . Moreover, the new Gale diagram will be k -neighborly whenever $\sum_{j=i}^{i+n-2} m_j \geq k+2$.

Proof. After such displacing, there will be lost

$$m_{i+n-1} \sum_{j=i+n+1}^{i-2} m_j$$

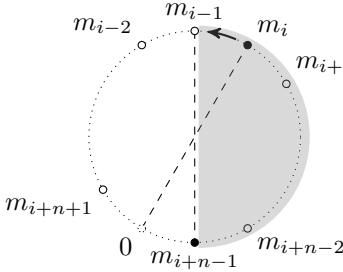


FIGURE 2. The displace operation. The sum of labels in the gray semicircle must be greater than $k + 2$ for preserving k -neighborliness.

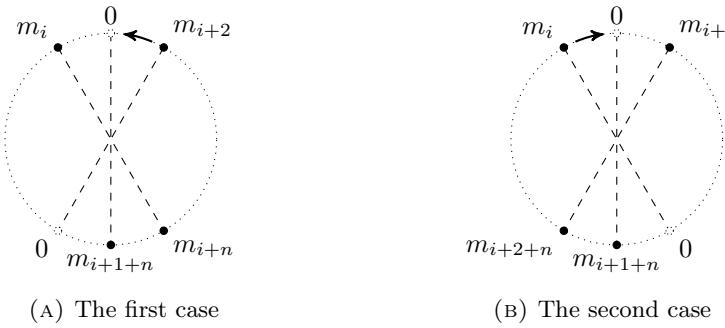


FIGURE 3. Two consecutive incomplete diameters

cofacets of the form $\{i, i + n - 1, j\}$, where $j \in [i + n + 1, i - 2]$. At the same time, there will appear m_{i+n-1} new cofacets of the form $\{i - 1, i + n - 1\}$. Note that

$$m_{i+n-1} > 0 \quad \text{and} \quad \sum_{j=i+n+1}^{i-2} m_j = \sum_{j=i+n}^{i-2} m_j \geq k + 1.$$

Therefore, after this operation, the total number of cofacets will be reduced at least by

$$m_{i+n-1}(k + 1 - 1) \geq k.$$

The displace operation reduces the sum $\sum_{j=i}^{i+n-2} m_j$ (the gray semicircle in the fig. 2) by 1. The sums in other semicircles are not reduced. If the inequality $\sum_{j=i}^{i+n-2} m_j \geq k + 2$ is fulfilled for the source diagram, then the displaced diagram is k -neighborly. \square

A diameter $\{m_i, m_{i+n}\}$ is called *complete* if $m_i m_{i+n} > 0$. Otherwise, $\{m_i, m_{i+n}\}$ is called *incomplete*. From Remark 9, it follows that an extremal reduced Gale diagram has at least one complete diameter.

Two diameters $\{m_i, m_{i+n}\}$ and $\{m_{i+1}, m_{i+1+n}\}$ are called *consecutive*.

Lemma 12. *An extremal k -neighborly Gale diagram has no consecutive incomplete diameters.*

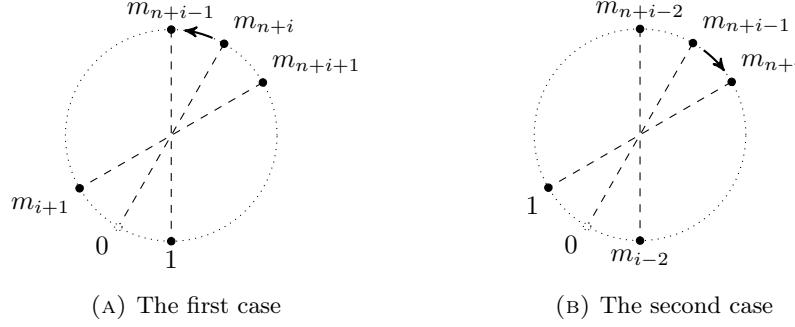


FIGURE 4. The sum of two consecutive labels cannot be less than 2.

Proof. Suppose to the contrary that some extremal k -neighborly Gale diagram D has consecutive incomplete diameters. From Remark 9, we know that D has at least one complete diameter. Hence there are three diameters $\{m_i, m_{i+n}\}$, $\{m_{i+1}, m_{i+1+n}\}$, $\{m_{i+2}, m_{i+2+n}\}$ such that one of the two conditions is satisfied (see fig. 3):

- (1) The first diameter is complete and the last two are incomplete.
- (2) The last diameter is complete and the first two are incomplete.

We examine only the first case. The second case is examined by analogy.

By the condition N,

$$\sum_{j=i+1}^{i+n-1} m_j \geq k+1.$$

Since $m_{i+1} = 0$ and $m_{i+n} > 0$, we get

$$\sum_{j=i+2}^{i+n} m_j \geq k+2.$$

From Lemma 11 (see also fig. 2), it follows that the displace operation

$$m_{i+2} := m_{i+2} - 1 \quad \text{and} \quad m_{i+1} := m_{i+1} + 1$$

reduces the total number of facets by at least k . Moreover, the new Gale diagram will be k -neighborly. Therefore, the source diagram D is not extremal. \square

Lemma 13. *For an extremal k -neighborly Gale diagram, $m_i + m_{i-1} \geq 2$, $i \in [2n]$.*

Proof. Suppose to the contrary that there is $i \in [2n]$ such that $m_i + m_{i-1} = 1$. (By the property P3, $m_i + m_{i-1} > 0$.) Hence one of two diameters $\{m_i, m_{n+i}\}$ and $\{m_{i-1}, m_{n+i-1}\}$ is incomplete. By Lemma 12, there are two symmetrical cases (see fig. 4):

- (1) $m_{i-1} = 1$, $m_i = 0$, $m_{i+1} > 0$, $m_{n+i-1} > 0$, $m_{n+i} > 0$, $m_{n+i+1} > 0$.
- (2) $m_{i-1} = 0$, $m_i = 1$, $m_{i-2} > 0$, $m_{n+i-2} > 0$, $m_{n+i-1} > 0$, $m_{n+i} > 0$.

We examine only the first case.

By the condition N,

$$\sum_{j=n+i+2}^{2n+i} m_j \geq k+1.$$

Since $m_{n+i} + m_{n+i+1} > m_{2n+i-1} + m_{2n+i}$, we get

$$\sum_{j=n+i}^{2n+i-2} m_j \geq k+2.$$

From Lemma 11, it follows that the displace operation

$$m_{n+i} := m_{n+i} - 1 \quad \text{and} \quad m_{n+i-1} := m_{n+i-1} + 1$$

reduces the total number of facets by at least k . Moreover, the new Gale diagram will be k -neighborly. Therefore, the source diagram D is not extremal. \square

Proposition 14. *Consider an extremal k -neighborly Gale diagram. Let $i \in [2n]$ and $q \geq 0$. If*

$$(14) \quad \sum_{j=i+1-t}^{i+n-1-t} m_j \geq m_i + q \quad \text{for every } t \in [n-1]$$

(in other words, in any open semicircle containing m_i , the sum of labels is not less than $m_i + q$), then $m_i \leq k+1-q$.

Proof. If $m_i > k+1-q$, then in any open semicircle containing m_i the sum of labels is greater than $k+1$. Thus the reducing $m_i := m_i - 1$ does not violate the property N. \square

Proposition 15. *For every extremal k -neighborly Gale diagram with n diameters,*

$$(15) \quad m_i \leq \begin{cases} k+5-n & \text{if } n \text{ is even,} \\ k+4-n & \text{if } n \text{ is odd,} \end{cases} \quad \text{and} \quad n \leq \begin{cases} k+2 & \text{if } k \text{ is even,} \\ k+3 & \text{if } k \text{ is odd.} \end{cases}$$

Proof. From Lemma 13, it follows that the inequality (14) holds for every $i \in [2n]$ and

$$q = \begin{cases} n-4 & \text{if } n \text{ is even,} \\ n-3 & \text{if } n \text{ is odd.} \end{cases}$$

By Proposition 14, we get $m_i \leq k+5-n$ for even n and $m_i \leq k+4-n$ for odd n .

This implies that $n \leq k+3$ for odd k .

Now suppose that k is even. Hence $n \leq k+4$. Note also that $m_i \leq 1$ for every $i \in [2n]$ and $n \in \{k+3, k+4\}$. By Lemma 13, we get $m_i = 1$ in this case. Thus, by removing 1 or 2 diameters, we can transform such a diagram to one in the example 2. Therefore, $n \leq k+2$ for an extremal Gale diagram if k is even. \square

Proposition 16. *For $k \in \{2, 3\}$, $\Delta_3(k) = (k+2)(k^2+k-3)/3$ and the best difference is attained in the example 2. For $k \in \{4, 5, 6\}$, $\Delta_3(k) = 2(k^2-1)$ and the best difference is attained in the example 1.*

Proof. The property N, Theorem 10, Lemma 13, and Proposition 15 impose strong restrictions. For small values of k , this allows us to find $\Delta_3(k)$ by using a computer program. For $k \leq 6$ it takes 1 minute. \square

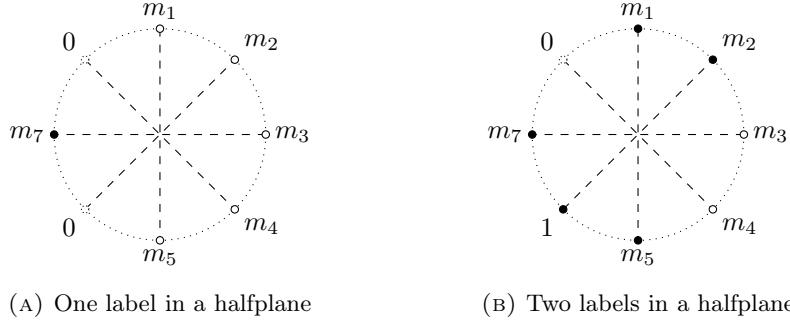


FIGURE 5. Two simple cases

4.3. The case $k \geq 6$. The goal of this subsection is to prove $\Delta_3(k) = 2(k^2 - 1)$ for $k \geq 6$. From Theorem 10, we know that the sum of labels of an extremal k -neighborly Gale diagram D is not greater than $4(k + 1)$. Thus, it is sufficient to show that the number of cofacets $f = f(D)$ cannot be less than

$$2(k^2 - 1) + 4(k + 1) = 2(k + 1)^2 \quad \text{for } k \geq 6.$$

(On the other hand, this value is attained in the example 1.)

We partition the task into three cases:

- (1) There is an open semicircle with only one positive label in our Gale diagram. See fig. 5a.
- (2) There is an open semicircle with exactly two positive labels and one of them has value 1. See fig. 5b.
- (3) In every open semicircle of the considered Gale diagram, there is at least two positive labels and if there are exactly two positive labels, then their values are not less than 2. This condition gives us possibility to use Proposition 14 with $q = 2$.

Lemma 17. *Let D be an extremal k -neighborly Gale diagram. If there is an open semicircle with only one positive label, then $f = f(D) \geq 2(k + 1)^2$ for $k \geq 2$.*

Proof. The general case is shown in the fig. 5a. By assumption, $m_6 = m_8 = 0$. For convenience, we use the notation

$$p = k + 1.$$

First note that $m_7 = p$, since the diagram is extremal and k -neighborly. Some of the other labels may equal zero and the number of diameters may be less than 4. By the property N,

$$(16) \quad m_1 + m_2 \geq p, \quad m_4 + m_5 \geq p, \quad m_2 + m_3 + m_4 \geq p.$$

Let us count the number of cofacets:

$$f = m_1 m_5 + m_3 m_7 + m_1 m_4 m_7 + m_2 m_4 m_7 + m_2 m_5 m_7.$$

Note that

$$f \geq m_1 m_4 m_7 + m_2 m_4 m_7 \geq m_4 p^2 \quad \text{and} \quad f \geq m_2 m_4 m_7 + m_2 m_5 m_7 \geq m_2 p^2.$$

Consequently, if $m_2 \geq 2$ or $m_4 \geq 2$, then $f \geq 2p^2 = 2(k + 1)^2$. It remains to analyze four cases:

- (1) If $m_2 = m_4 = 0$, then, using (16), we have $m_1 \geq p$, $m_5 \geq p$, $m_3 \geq p$. Hence $f = m_1 m_5 + m_3 m_7 \geq 2p^2$.
- (2) If $m_2 = 0$ and $m_4 = 1$, then $m_1 \geq p$, $m_5 \geq p - 1$, $m_3 \geq p - 1$. Hence $f \geq p(p - 1) + (p - 1)p + p^2 = 3p^2 - 2p \geq 2p^2$ for $p \geq 2$.
- (3) The case $m_2 = 1$ and $m_4 = 0$ is analysed by analogy with the previous one.
- (4) If $m_2 = m_4 = 1$, then $m_1 \geq p - 1$, $m_5 \geq p - 1$, $m_3 \geq p - 2$. Hence $f \geq (p - 1)^2 + (p - 2)p + (p - 1)p + p + (p - 1)p = 2p^2 + 2p^2 - 5p + 1 > 2p^2$ for $p \geq 3$. \square

Lemma 18. *Let D be an extremal k -neighborly Gale diagram. If there is an open semicircle with exactly two positive labels and the value of one of them is equal to 1, then $f = f(D) \geq 2(k + 1)^2$ for $k \geq 4$.*

Proof. By Lemma 13, the sum of two consecutive labels cannot be less than 2. Consequently, the neighbors of the label with value 1 must have positive values. Therefore, the number of diameters of the diagram cannot be greater than 4. The general case is shown in the fig. 5b. By assumption, $m_8 = 0$.

Here, we do not need to consider the cases that was analyzed in Lemma 17. Hence

$$(17) \quad m_1 > 0, \quad m_2 > 0$$

(the corresponding points are painted in fig. 5b). By Lemma 13,

$$(18) \quad m_5 > 0.$$

Let $p = k + 1$. By the property N,

$$(19) \quad m_7 + 1 \geq p,$$

$$(20) \quad m_1 + m_2 \geq p,$$

$$(21) \quad m_2 + m_3 + m_4 \geq p,$$

$$(22) \quad m_3 + m_4 + m_5 \geq p,$$

$$(23) \quad m_4 + m_5 + 1 \geq p.$$

Let us count the number of cofacets:

$$f = m_2 + m_3 m_7 + m_2 m_5 m_7 + m_4 m_7 (m_1 + m_2) + m_1 (m_3 + m_4 + m_5).$$

In particular, by using (19), (23), we obtain

$$(24) \quad \begin{aligned} f &\geq m_3 m_7 + m_2 m_7 (m_4 + m_5) + m_1 (m_3 + m_4 + m_5) \\ &\geq m_3 (p - 1) + m_2 (p - 1)^2 + m_1 (m_3 + m_4 + m_5). \end{aligned}$$

Further proof is reduced to an analysis of all possible cases regarding the values of labels m_4 and m_2 . In all cases, we get $f \geq 2p^2$ for $p \geq 5$.

m₄ ≥ 2. By the inequalities (18), (19), (20), (22), we have

$$\begin{aligned} f &\geq m_2 (1 + m_5 m_7) + m_4 m_7 (m_1 + m_2) + m_1 (m_3 + m_4 + m_5) \\ &\geq m_2 p + 2(p - 1)p + m_1 p \geq 3p^2 - 2p \geq 2p^2. \end{aligned}$$

m₂ ≥ 3. By using (24), (17), (20), (22), (23), we obtain

$$f \geq 3(p - 1)(p - 1) + p \geq 2p^2.$$

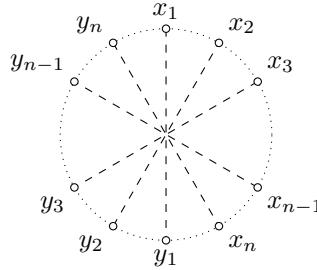


FIGURE 6. The induction step

m₂ = 2 and m₄ ≤ 1. Hence $m_1 \geq p - 2$, $m_3 \geq p - 3$, $m_5 \geq p - 2$, and

$$f \geq (p-3)(p-1) + 2(p-1)^2 + (p-2)(p-3+p-1) \geq 2p^2.$$

m₂ = 1 and m₄ ≤ 1. This implies that $m_1 \geq p - 1$, $m_3 \geq p - 2$, $m_5 \geq p - 2$,

$$f \geq (p-2)(p-1) + (p-1)^2 + (p-1)(p-2+p-1) \geq 2p^2. \quad \square$$

Lemma 19. *Let D be an extremal k -neighborly Gale diagram. Suppose that in every open semicircle, there is at least two positive labels, and if there are exactly two positive labels, then their values are not less than 2. Then $f(D) \geq 2(k+1)^2$ for $k \geq 5$.*

Proof. The proof is by induction over k . The case $k = 5$ is covered by Proposition 16.

Suppose that the statement of lemma is true for $k = t$, $t \geq 5$.

Let us consider an extremal $(t+1)$ -neighborly Gale diagram D with labels $\{m_0, \dots, m_{2n-1}\}$. By Remark 9, the diagram has at least one complete diameter $\{m_{j^*}, m_{n+j^*}\}$. Reduce m_{j^*} and m_{n+j^*} by 1. Obviously, the new diagram D' is at least t -neighborly. For the labels of D' we will use the following notations (see fig. 6):

$$\begin{aligned} x_1 &= m_{j^*} - 1, & y_1 &= m_{n+j^*} - 1, \\ x_i &= m_{j^*+i-1}, & y_i &= m_{n+j^*+i-1}, \quad i \in [2, n]. \end{aligned}$$

Observe that

$$(25) \quad \sum_{i=2}^n x_i \geq t+2 \quad \text{and} \quad \sum_{i=2}^n y_i \geq t+2.$$

By the induction hypothesis, $f(D') \geq 2(t+1)^2$. Hence it is sufficient to prove that

$$f(D) - f(D') \geq 4(t+1) + 2.$$

By assumption, the source diagram D satisfies the conditions of Proposition 14 with $q = 2$. Thus

$$m_i \leq t+2-2, \quad \forall i \in [2n],$$

and

$$(26) \quad x_j \leq t, \quad y_j \leq t, \quad \forall j \in [n].$$

Let us count the difference between $f(D)$ and $f(D')$. We start from the cofacets with exactly two points. The diagram D has exactly $x_1 + y_1 + 1$ such cofacets that

do not present in D' . When reducing m_{j^*} and m_{n+j^*} , there are also lost triangular cofacets of the following types:

- 1) $\{j^*, r, s\}$, where $r \in [j^* + 2, j^* + n - 1]$ and $s \in [j^* + n + 1, r + n - 1]$.
- 2) $\{j^* + n, r, s\}$, where $r \in [j^* + 1, j^* + n - 2]$ and $s \in [r + n + 1, j^* + 2n - 1]$.

In total, we get

$$(27) \quad f(D) - f(D') = \left(\sum_{i=2}^n x_i \right) \left(\sum_{i=2}^n y_i \right) - \sum_{i=2}^n x_i y_i + x_1 + y_1 + 1.$$

Let us suppose that $\min\{x_i, y_i\} \leq 2$ for every $i \in [2, n]$. Hence

$$\sum_{i=2}^n x_i y_i \leq 2 \left(\sum_{i=2}^n y_i \right) + 2 \left(\sum_{i=2}^n x_i \right) - 4.$$

Combining this with (25), we obtain

$$\begin{aligned} f(D) - f(D') &= \left(\sum_{i=2}^n x_i - 2 \right) \left(\sum_{i=2}^n y_i - 2 \right) + x_1 + y_1 + 1 \\ &\geq t^2 + 1 \geq 4(t + 1) + 2 \quad \text{for } t \geq 5. \end{aligned}$$

Now we suppose that

$$\max_{i \in [2, n]} (\min\{x_i, y_i\}) \geq 3.$$

Hence $x_{i'} \geq 3$ and $y_{i'} \geq 3$ for some $i' \in [2, n]$. In addition, $x_{i'} \leq t$ and $y_{i'} \leq t$ by (26). Let us denote

$$S(x_{i'}, y_{i'}) = f(D) - f(D'), \quad \text{where } x_{i'} \in [3, t], y_{i'} \in [3, t].$$

Let $I = [n] \setminus \{1, i'\}$ and

$$G(x_{i'}, y_{i'}) = \left(\sum_{i \in I} x_i \right) \left(\sum_{i \in I} y_i \right) - \sum_{i \in I} x_i y_i.$$

It is clear that $G(x_{i'}, y_{i'}) \geq 0$. Combining this with (27), we get

$$\begin{aligned} S(x_{i'}, y_{i'}) &= G(x_{i'}, y_{i'}) + y_{i'} \sum_{i \in I} x_i + x_{i'} \sum_{i \in I} y_i + x_1 + y_1 + 1 \\ &\geq y_{i'} \sum_{i \in I} x_i + x_{i'} \sum_{i \in I} y_i + x_1 + y_1 + 1. \end{aligned}$$

Since the diagram D' is t -neighborly, we have

$$\sum_{i=1}^n (x_i + y_i) - x_{i'} - y_{i'} \geq 2(t + 1).$$

Thus

$$x_1 + y_1 \geq 2t + 2 - \sum_{i \in I} (x_i + y_i)$$

and

$$S(x_{i'}, y_{i'}) \geq (y_{i'} - 1) \sum_{i \in I} x_i + (x_{i'} - 1) \sum_{i \in I} y_i + 2t + 3.$$

From (25), we have

$$\sum_{i \in I} x_i \geq t + 2 - x_{i'} \quad \text{and} \quad \sum_{i \in I} y_i \geq t + 2 - y_{i'}.$$

Hence

$$S(x_{i'}, y_{i'}) \geq (y_{i'} - 1)(t + 2 - x_{i'}) + (x_{i'} - 1)(t + 2 - y_{i'}) + 2t + 3.$$

For the sake of convenience, we use the notations:

$$p = t + 1, \quad x = x_{i'} - 1, \quad y = y_{i'} - 1.$$

Hence $p \geq 6$ (by assumption), $x \in [2, p - 2]$, $y \in [2, p - 2]$, and

$$S(x_{i'}, y_{i'}) = S(x + 1, y + 1) \geq y(p - x) + x(p - y) + 2p + 1.$$

But

$$y(p - x) + x(p - y) \geq 4(p - 2) \quad \text{for } x \in [2, p - 2], \quad y \in [2, p - 2].$$

Therefore,

$$S(x_{i'}, y_{i'}) \geq 4p + 2 + 2p - 9 \geq 4p + 2 = 4(t + 1) + 2 \quad \text{if } t \geq 5. \quad \square$$

ACKNOWLEDGEMENTS

The author is grateful to Günter Ziegler for valuable advices and to Arnau Padrol for useful discussions.

REFERENCES

- [1] D. Barnette. *The minimum number of vertices of a simple polytope*. Israel J. Math. **10**, 1971, 121–125.
- [2] G. Blind and R. Blind. *Convex polytopes without triangular faces*. Isr. J. Math. **71**, 1990, 129–134.
- [3] É. Fusy. *Counting d-Polytopes with d + 3 Vertices*. Electr. J. Comb. **13**(1), 2006, research paper R23.
- [4] M. Henk, J. Richter-Gebert and G. Ziegler. *Basic properties of convex polytopes*. In J.E. Goodman and J. O'Rourke, editors, *Handbook of Discrete and Computational Geometry*. Chapman & Hall/CRC Press, Boca Raton, 2nd edition, 2004, 355–382.
- [5] R. Gillmann. *0/1-Polytopes: Typical and Extremal Properties*. PhD Thesis, TU Berlin, 2006.
- [6] B. Grünbaum. *Convex polytopes*, 2nd edition (V. Kaibel, V. Klee and G.M. Ziegler, eds.), Springer, 2003.
- [7] A. Maksimenko. *k-Neighborly Faces of the Boolean Quadric Polytopes*. Journal of Mathematical Sciences. **203**(6), 2014, 816–822.
- [8] A. Maksimenko. *A special role of Boolean quadratic polytopes among other combinatorial polytopes*. arXiv:1408.0948
- [9] D.A. Marcus. *Minimal Positive 2-Spanning Sets of Vectors*. Proc. AMS, **82**(2), 1981, 165–172.
- [10] P. McMullen. *The maximum numbers of faces of a convex polytope*. Mathematika, **17**, 1970, 179–184.
- [11] G.M. Ziegler. *Lectures on Polytopes*, Springer, 1995.

LABORATORY OF DISCRETE AND COMPUTATIONAL GEOMETRY, P.G. DEMIDOV YAROSLAVL STATE UNIVERSITY, UL. SOVETSKAYA 14, YAROSLAVL 150000, RUSSIA
E-mail address: maximenko.a.n@gmail.com